RELATIONSHIPS AMONG CLASSES OF SPHERICAL MATRIX DISTRIBUTIONS

TECHNICAL REPORT NO. 10

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INSTITUTE OF APPLIED MATHEMATICS, ACADEMIA SINICA
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U.S. ARMY RESEARCH OFFICE
CONTRACT DAAG 29-82-K-0156

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1. Introduction.

The theory of multivariate analysis has been based mainly on the normal population. Statisticians have been trying to extend the sample theory in multivariate analysis to cases of the observations being not necessarily either normal or independent. In the last decade, especially the last five years, many statisticians have been interested in a specific class of distributions, one of elliptically contoured distributions, and found that this class has many properties similar to the normal distribution.

If the c.f. (charcteristic function) of an n-dimensional random vector $\underset{\sim}{x}$ has the form $\exp(i\underset{\sim}{t}'\underset{\sim}{\mu})\phi(\underset{\sim}{t}'\underset{\sim}{\Sigma}\underset{\sim}{t})$, where $\underset{\sim}{\mu}$: $n \times 1$, $\underset{\sim}{\Sigma}$: $n \times n$ and $\underset{\sim}{\Sigma} \geq 0$, we say that $\underset{\sim}{x}$ has an elliptically contoured distribution with parameters $\underset{\sim}{\mu}$, $\underset{\sim}{\Sigma}$, and ϕ, and write $\underset{\sim}{x} \sim EC_n(\underset{\sim}{\mu},\underset{\sim}{\Sigma},\phi)$. When $\underset{\sim}{\mu} = \underset{\sim}{0}$ and $\underset{\sim}{\Sigma} = \underset{\sim}{I}_n$, we call $EC_n(\underset{\sim}{0},\underset{\sim}{I}_n,\phi)$ a spherical distribution and write $\underset{\sim}{x} \sim S_n(\phi)$, because $\underset{\sim}{x} \sim S_n(\phi)$ iff $\underset{\sim}{x} \overset{d}{=} \underset{\sim}{\Gamma}\underset{\sim}{x}$ for each $\underset{\sim}{\Gamma} \in \underset{\sim}{O}(n)$, where $\underset{\sim}{O}(n)$ is the set of $n \times n$ orthogonal matrices and the notation "$\underset{\sim}{x} \overset{d}{=} \underset{\sim}{y}$" means that $\underset{\sim}{x}$ and $\underset{\sim}{y}$ have the same distribution.

As an extension of multivariate normal sampling theory, several classes of spherical matrix distributions are defined and have been discussed by many authors. Here are the main three classes.

Definition 1. Let $\underset{\sim}{X} = (\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_p)$ be an $n \times p$ random matrix.

(1) $F_1 = \{\underset{\sim}{X}: \underset{\sim}{\Gamma}\underset{\sim}{X} \overset{d}{=} \underset{\sim}{X}$ for every $\Gamma \in \underset{\sim}{O}(n)\}$. The $\underset{\sim}{X}$ in F_1 is called left-spherical by Dawid (1977).

(2) $F_2 = \{\underset{\sim}{X}: (\underset{\sim}{\Gamma}_1\underset{\sim}{x}_1,\ldots,\underset{\sim}{\Gamma}_p\underset{\sim}{x}_p) \overset{d}{=} (\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_p)$ for every $\underset{\sim}{\Gamma}_i \in \underset{\sim}{O}(n)$, $i = 1,\ldots,p\}$.

(3) $F_3 = \{\underset{\sim}{X}: \underset{\sim}{\Gamma}(\mathrm{vec}\,\underset{\sim}{X}) \overset{d}{=} \mathrm{vec}\,\underset{\sim}{X}$ for every $\underset{\sim}{\Gamma} \in \underset{\sim}{O}(np)\}$. Each of F_1, F_2, and F_3 is called a class of spherical matrix distributions.

These classes contain many important distributions, such as the multivariate normal distributions, the multivariate t-distributions, the multivariate Beta-distributions and the multivariate stable laws. The class F_1 was studied by Dawid (1977, 1978), Fraser and Ng (1980), Jensen and Good (1981), and Kariya (1981a and b). The class F_2 was defined by Anderson and Fang (1982b), and the class F_3 was discussed by Chmielewski (1980), Kariya (1981a), Jensen and Good (1981) and Anderson and Fang (1982b and c). They have found that many statistics are invariant in these classes. However the relationships among these classes are not yet very clear. Therefore it may be valuable to consider the relationships among them.

Throughout this paper, $\underset{\sim}{X} = (\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_p) = (\underset{\sim}{x}_{(1)}, \ldots, \underset{\sim}{x}_{(n)})' = (x_{ij})$ denotes an $n \times p$ random matrix with $n \geq p$, $\underset{\sim}{I}_n$ denotes the $n \times n$ identity matrix, $\mathrm{diag}(a_1, \ldots, a_n)$ denotes an $n \times n$ diagonal matrix with diagonal elements $a_1, \ldots, a_n$, $\underset{\sim}{A}'$, $\mathrm{rk}\underset{\sim}{A}$ and $\mathrm{tr}\underset{\sim}{A}$ denote the transpose of $\underset{\sim}{A}$, the rank of $\underset{\sim}{A}$ and the trace of $\underset{\sim}{A}$, $\underset{\sim}{u}^{(n)}$ denotes a random vector which is uniformly distributed on the unit sphere in R^n, and $\Omega_n(\underset{\sim}{t}'\underset{\sim}{t})$ denotes its c.f.

In Section 2 some basic properties about them are listed. Sections 3 and 4 are the main part of the paper. In the last section we summarize the invariant distributions in these classes as a table for applications.

2. Preliminary.

In this section we recall some basic properties of F_1, F_2, and F_3 which will be used frequently in this paper. The following lemmas and Theorem 1 are from Dawid (1977), Kariya (1981), and Anderson and Fang (1982b).

Let $\underset{\sim}{T} = (\underset{\sim}{t}_1,\ldots,\underset{\sim}{t}_p)$ be an $n \times p$ matrix.

Lemma 1.

(1) $\underset{\sim}{X} \varepsilon F_1$ iff the c.f. of $\underset{\sim}{X}$ has the form $\phi(\underset{\sim}{T}'\underset{\sim}{T})$;

(2) $\underset{\sim}{X} \varepsilon F_2$ iff the c.f. of $\underset{\sim}{X}$ has the form $\phi(\underset{\sim}{t}_1'\underset{\sim}{t}_1,\ldots,\underset{\sim}{t}_p'\underset{\sim}{t}_p)$;

(3) $\underset{\sim}{X} \varepsilon F_3$ iff the c.f. of $\underset{\sim}{X}$ has the form $\phi(\mathrm{tr}\underset{\sim}{T}'\underset{\sim}{T})$.

Lemma 2. Suppose $\underset{\sim}{X}$ has a pdf (probability density function). Then

(1) $\underset{\sim}{X} \varepsilon F_1$ iff the pdf of $\underset{\sim}{X}$ has the form $f(\underset{\sim}{X}'\underset{\sim}{X})$;

(2) $\underset{\sim}{X} \varepsilon F_2$ iff the pdf of $\underset{\sim}{X}$ has the form $f(\underset{\sim}{x}_1'\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_p'\underset{\sim}{x}_p)$;

(3) $\underset{\sim}{X} \varepsilon F_3$ iff the pdf of $\underset{\sim}{X}$ has the form $f(\mathrm{tr}\underset{\sim}{X}'\underset{\sim}{X})$.

Theorem 1.

(a) $\underset{\sim}{X} \varepsilon F_1$ if $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{U}_1\underset{\sim}{A}$, where $\underset{\sim}{A}$ and $\underset{\sim}{U}_1$ are independent, $\underset{\sim}{U}_1 \varepsilon F_1$ and $\underset{\sim}{U}_1'\underset{\sim}{U}_1 = \underset{\sim}{I}_p$;

(b) $\underset{\sim}{X} \varepsilon F_2$ iff $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{U}_2\underset{\sim}{R}$, where $\underset{\sim}{R} = \mathrm{diag}(R_1,\ldots,R_p)$ and $\underset{\sim}{U}_2 = (\underset{\sim}{u}_1,\ldots,\underset{\sim}{u}_p)$ are independent, $R_i \geq 0$, $i = 1,\ldots,p$, and $\underset{\sim}{u}_1,\ldots,\underset{\sim}{u}_p$ i.i.d. $\underset{\sim}{u}_1 \overset{d}{=} \underset{\sim}{u}^{(n)}$;

(c) $\underset{\sim}{X} \varepsilon F_3$ iff $\underset{\sim}{X} \overset{d}{=} R\underset{\sim}{U}_3$, where $R \geq 0$ and $\underset{\sim}{U}_3$ are independent, $\mathrm{vec}\,\underset{\sim}{U}_3 \overset{d}{=} \underset{\sim}{u}^{(np)}$.

The distribution of $\underset{\sim}{U}_1$ is called the uniform distribution. The matrices $\underset{\sim}{U}_1$, $\underset{\sim}{U}_2$ and $\underset{\sim}{U}_3$ play roles of coordinate systems in F_1, F_2 and F_3, respectively. In this paper when we write $\underset{\sim}{X} \overset{d}{=} R\underset{\sim}{U}_3$, $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{U}_2\underset{\sim}{R}$, and $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{U}_1\underset{\sim}{A}$, they have the meaning in Theorem 1 unless we make another explanation.

Lemma 3. The distribution of $\underset{\sim}{X}$ is fully determined by that of $\underset{\sim}{X}'\underset{\sim}{X}$ if $\underset{\sim}{X} \varepsilon F_1$, by that of $(\underset{\sim}{x}_1'\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_p'\underset{\sim}{x}_p)$ if $\underset{\sim}{X} \varepsilon F_2$, and by that of $\mathrm{tr}\underset{\sim}{X}'\underset{\sim}{X}$ if $\underset{\sim}{X} \varepsilon F_3$.

Lemma 3 shows us that if $\underset{\sim}{X} \varepsilon F_1$, $\underset{\sim}{Y} \varepsilon F_1$, and $\underset{\sim}{X}'\underset{\sim}{X} \overset{d}{=} \underset{\sim}{Y}'\underset{\sim}{Y}$, then $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{Y}$. Similar statements hold for F_2 and F_3.

The following properties of the operation "$\overset{d}{=}$" are given by Anderson and Fang (1982a):

(1) If $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{Y}$ and $f_i(\cdot)$, $i = 1,\ldots,m$, are Borel functions, then $(f_1(\underset{\sim}{X}),\ldots,f_m(\underset{\sim}{X})) \overset{d}{=} (f_1(\underset{\sim}{Y}),\ldots,f_m(\underset{\sim}{Y}))$.

(2) If z is independent of $\underset{\sim}{X}$ and $\underset{\sim}{Y}$, respectively, then

(a) $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{Y}$ implies $z\underset{\sim}{X} \overset{d}{=} z\underset{\sim}{Y}$;

(b) if $P(z > 0) = 1$ and the c.f. of $\log z$ $\phi_{\log z}(t) \neq 0$ for almost all t then $z\underset{\sim}{X} \overset{d}{=} z\underset{\sim}{Y}$ implies $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{Y}$.

Lemma 4. Suppose $\underset{\sim}{X} \varepsilon F_1$ and $P(|\underset{\sim}{X}'\underset{\sim}{X}|=0) = 0$, then

(1) $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{U}_1\underset{\sim}{A} \overset{d}{=} \underset{\sim}{U}_1\underset{\sim}{B}$ implies $\underset{\sim}{A} \overset{d}{=} \underset{\sim}{B}$, where $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are two upper triangular matrices with positive diagonal elements;

(2) $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{Q}\underset{\sim}{T}$, where $\underset{\sim}{Q}'\underset{\sim}{Q} = \underset{\sim}{I}_p$ and $\underset{\sim}{T}$ is an upper triangular matrix with positive diagonal elements, implies $\underset{\sim}{Q} \overset{d}{=} \underset{\sim}{U}_1$ and $\underset{\sim}{Q}$ is independent of $\underset{\sim}{T}$.

Proof. The existence of $\underset{\sim}{A}$ in (1) follows from the argument of Dawid (1977). We consider mapping $f\colon \underset{\sim}{A} \to f(\underset{\sim}{A}) = \underset{\sim}{A}'\underset{\sim}{A}$, where $\underset{\sim}{A}$ is an upper triangular matrix with positive diagonal elements. Clearly, f is a one-to-one mapping. As $\underset{\sim}{A}'\underset{\sim}{A} \overset{d}{=} \underset{\sim}{B}'\underset{\sim}{B}$, then (1) follows from $E[h(\underset{\sim}{A})] = E[h(f^{-1}(\underset{\sim}{A}'\underset{\sim}{A}))] = E[h(f^{-1}(\underset{\sim}{B}'\underset{\sim}{B}))] = E[h(\underset{\sim}{B})]$ for each Borel

function $h \geq 0$. Note that if $|\underset{\sim}{X}'\underset{\sim}{X}| \neq 0$, there is a unique decomposition $\underset{\sim}{X} = \underset{\sim}{Q}\underset{\sim}{T}$, where $\underset{\sim}{Q}'\underset{\sim}{Q} = \underset{\sim}{I}_p$ and $\underset{\sim}{T}$ is an upper triangular matrix with positive diagonal elements. Let the function $g(\underset{\sim}{X}) = (\underset{\sim}{Q},\underset{\sim}{T})$, we have $(\underset{\sim}{Q},\underset{\sim}{T}) = g(\underset{\sim}{X}) \overset{d}{=} g(\underset{\sim}{U}_1\underset{\sim}{A}) = (\underset{\sim}{U}_1,\underset{\sim}{A})$ as $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{U}_1\underset{\sim}{A}$, which completes the proof. Q.E.D.

From now on when we write $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{U}_1\underset{\sim}{A}$ for $\underset{\sim}{X} \in \mathcal{F}_1$, we always consider that $\underset{\sim}{A}$ is an upper triangular matrix with nonnegative diagonal elements.

3. Relationships among $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$.

By Definition 1, clearly $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \mathcal{F}_3$. But how much are the differences among them at all? In this section, we will discuss them in the following aspects: the coordinate system and the coordinate transformations, marginal distributions, marginal densities and sphericity. First of all, start with $\underset{\sim}{U}_1$, $\underset{\sim}{U}_2$ and $\underset{\sim}{U}_3$.

Lemma 5. Suppose $\underset{\sim}{X} \in \mathcal{F}_1$ and $P(\underset{\sim}{x}_i = \underset{\sim}{0}) = 0$, $i = 1,\ldots,p$. Then $\underset{\sim}{X} \in \mathcal{F}_2$ iff $\underset{\sim}{X}$ satisfies the following conditions:

(1) $\underset{\sim}{x}_1/\|\underset{\sim}{x}_1\|,\ldots,\underset{\sim}{x}_p/\|\underset{\sim}{x}_p\|$ are independent; and

(2) $(\|\underset{\sim}{x}_1\|,\ldots,\|\underset{\sim}{x}_p\|)$ and $(\underset{\sim}{x}_1/\|\underset{\sim}{x}_1\|,\ldots,\underset{\sim}{x}_p/\|\underset{\sim}{x}_p\|)$ are independent.

Proof. The assertion follows from Anderson and Fang (1982b). Q.E.D.

Corollary 1. $\underset{\sim}{U}_1 \notin \mathcal{F}_2$.

Proof. As $\underset{\sim}{U}_1'\underset{\sim}{U}_1 = \underset{\sim}{I}_p$ and $\underset{\sim}{U}_1 \in \mathcal{F}_1$, $\underset{\sim}{U}_1 = (\underset{\sim}{u}_1,\ldots,\underset{\sim}{u}_p)$ where $\underset{\sim}{u}_1,\ldots,\underset{\sim}{u}_p$ are not independent, the corollary follows from Lemma 5. Q.E.D.

Lemma 6. $\underset{\sim}{U}_2 \notin \mathcal{F}_3$.

Proof. Suppose $\underset{\sim}{U}_2 = (\underset{\sim}{u}_1,\ldots,\underset{\sim}{u}_p) = (\underset{\sim}{u}_{(1)},\ldots,\underset{\sim}{u}_{(n)})' \varepsilon \mathcal{F}_3$, then $\underset{\sim}{U}_2 \overset{d}{=} R\underset{\sim}{U}_3$ for some $R \geq 0$ being independent of $\underset{\sim}{U}_3$. As $\underset{\sim}{u}_1,\ldots,\underset{\sim}{u}_p$ are independent and $\underset{\sim}{u}_{(i)}$ $(i = 1,\ldots,n)$ has a spherical distribution, the distribution of $\underset{\sim}{u}_{(i)}$ must be normal and the distribution of $\underset{\sim}{U}_2$ must be normal by Kelker (1970). The contradiction proves the theorem. Q.E.D.

The following example shows us that the condition (2) in Lemma 5 is necessary, and its distribution belongs to $\mathcal{F}_1$, but not to $\mathcal{F}_2$.

Example 1. Let $\underset{\sim}{X}$ be an $n \times p$ random matrix with a pdf $f(\underset{\sim}{X}) = c|\underset{\sim}{I}_p+\underset{\sim}{X}'\underset{\sim}{X}|^{-\frac{1}{2}(n+p)}$, where c is a constant. We want to prove that $(\|\underset{\sim}{x}_1\|,\ldots,\|\underset{\sim}{x}_p\|) \equiv (R_1,\ldots,R_p)$ and $(\underset{\sim}{x}_1/\|\underset{\sim}{x}_1\|,\ldots,\underset{\sim}{x}_p/\|\underset{\sim}{x}_p\|) \equiv \underset{\sim}{U}$ are not independent. It is easy to see that if $\underset{\sim}{R} = \text{diag}(R_1,\ldots,R_p)$ and $\underset{\sim}{U}$ are independent, then $f(\underset{\sim}{X}) = f(\underset{\sim}{U}\underset{\sim}{R})$ can be written in the form of $f(\underset{\sim}{X}) = g_1(\underset{\sim}{R})g_2(\underset{\sim}{U})$; similarly $|\underset{\sim}{I}_p+\underset{\sim}{X}'\underset{\sim}{X}| = h_1(\underset{\sim}{R})h_2(\underset{\sim}{U})$ for some two functions h_1 and h_2. Take $\underset{\sim}{R} = \underset{\sim}{I}_p$ and $\underset{\sim}{U}'\underset{\sim}{U} = \underset{\sim}{I}_p$, respectively, to show

$$|\underset{\sim}{I}_p+\underset{\sim}{X}'\underset{\sim}{X}| = |\underset{\sim}{I}_p+\underset{\sim}{R}\underset{\sim}{U}'\underset{\sim}{U}\underset{\sim}{R}| = k|\underset{\sim}{I}_p+\underset{\sim}{R}^2||\underset{\sim}{I}_p+\underset{\sim}{U}'\underset{\sim}{U}| ,$$

for each $\underset{\sim}{R} = \text{diag}(R_1,\ldots,R_p)$ with $R_1 > 0,\ldots,R_p > 0$, $\underset{\sim}{U} = (\underset{\sim}{u}_1,\ldots,\underset{\sim}{u}_p)$, and $\underset{\sim}{u}_i'\underset{\sim}{u}_i = 1$, $i = 1,\ldots,p$, where k is a constant. Let $\underset{\sim}{R} = t^{-1}\underset{\sim}{I}_p$, $t > 0$, we have

$$k|t^2 \underset{\sim}{I}_p + \underset{\sim}{I}_p||\underset{\sim}{I}_p + \underset{\sim}{U}'\underset{\sim}{U}| = |t^2 \underset{\sim}{I}_p + \underset{\sim}{U}'\underset{\sim}{U}|, \quad \text{for each} \quad t > 0,$$

i.e.

$$k(1+t^2)^p \prod_{i=1}^{p} (1+\lambda_i) = \prod_{i=1}^{p} (t^2+\lambda_i), \quad \text{for each} \quad t > 0,$$

where $\lambda_1,\ldots,\lambda_p$ are the eigenvalues of $\underset{\sim}{U}'\underset{\sim}{U}$. Obviously, this is impossible. This contradiction reveals that $\underset{\sim}{R}$ and $\underset{\sim}{U}$ are not independent.

3.1 Coordinate transformations.

In this paragraph we try to give the stochastic representations for F_1, F_2 and F_3 under the same "coordinate system". Maybe they can help us to understand these classes more clearly. The generalized Dirichlet distribution defined by Anderson and Fang (1982a) will be used. Consider a random vector $(z_1,\ldots,z_m)'$ such that $(z_1,\ldots,z_m) \overset{d}{=} R^2(d_1,\ldots,d_m)$, where $0 \le R \sim F(\cdot)$, R is independent of $(d_1,\ldots,d_m)$, $d_1 + \cdots + d_m = 1$, and $(d_1,\ldots,d_{m-1}) \sim D_m(\alpha_1,\ldots,\alpha_{m-1};\alpha_m)$ (Dirichlet distribution) with $\alpha_1,\ldots,\alpha_m > 0$ and $n = 2(\alpha_1 + \cdots + \alpha_m)$ is an integer; we write $(z_1,\ldots,z_{m-1}) \sim G_m(\alpha_1,\ldots,\alpha_{m-1};\alpha_m;F)$ or $(z_1,\ldots,z_m) \sim G_m(\alpha_1,\ldots,\alpha_m;F)$.

Anderson and Fang (1982b) pointed out that if $\underset{\sim}{U}_3 \overset{d}{=} \underset{\sim}{U}_2\underset{\sim}{R}$ with $\underset{\sim}{R} = \mathrm{diag}(R_1,\ldots,R_p)$, then $(R_1^2,\ldots,R_{p-1}^2) \sim D_p(\frac{1}{2}n,\ldots,\frac{1}{2}n;\frac{1}{2}n)$ and $R_1^2 + \cdots + R_p^2 = 1$. Also $\underset{\sim}{X} \overset{d}{=} R\underset{\sim}{U}_3 \in F_3$ and $R \sim F(\cdot)$ iff $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{U}_2\underset{\sim}{R}$ with $\underset{\sim}{R}^2 = \mathrm{diag}(R_1^2,\ldots,R_p^2) \sim G_p(\frac{1}{2}n,\ldots,\frac{1}{2}n;F)$.

Let $\underset{\sim}{Y} = (y_{ij}) = (\underset{\sim}{y}_1,\ldots,\underset{\sim}{y}_p)$ be an $n \times p$ random matrix with i.i.d. elements and $y_{ij} \sim N(0,1)$. There exists an upper triangular matrix $T = (t_{ij})$ with positive diagonal elements such that $\underset{\sim}{T}'\underset{\sim}{T} = \underset{\sim}{Y}'\underset{\sim}{Y}$.

This is the famous Bartlett decomposition. It is a well-known fact that $\{t_{ij},\ 1 \le i \le j \le p\}$ are independent, $t_{ij} \sim N(0,1)$ for $i < j$ and $t_{ii}^2 \sim \chi^2_{n-i+1}$, $i = 1,\ldots,p$. Let $\underset{\sim}{t}_j = (t_{1j},\ldots,t_{jj})'$, $j = 1,\ldots,p$. Then we have

<u>Theorem 2</u>. Suppose $\underset{\sim}{U}_2 \overset{d}{=} \underset{\sim}{U}_1\underset{\sim}{A}$, where $\underset{\sim}{A}$ is an upper triangular matrix with positive diagonal elements. Then $\underset{\sim}{A} \overset{d}{=} \underset{\sim}{T}\underset{\sim}{R}^{-1}$, where $\underset{\sim}{R} = \mathrm{diag}(\|\underset{\sim}{t}_1\|,\ldots,\|\underset{\sim}{t}_p\|)$.

<u>Proof</u>. As $(\underset{\sim}{y}_1/\|\underset{\sim}{y}_1\|,\ldots,\underset{\sim}{y}_p/\|\underset{\sim}{y}_p\|) \overset{d}{=} \underset{\sim}{U}_2 \overset{d}{=} \underset{\sim}{U}_1\underset{\sim}{A}$ and $\underset{\sim}{Y}'\underset{\sim}{Y} = \underset{\sim}{T}'\underset{\sim}{T}$, we have $\underset{\sim}{y}_i'\underset{\sim}{y}_i = \underset{\sim}{t}_i'\underset{\sim}{t}_i$, $i = 1,\ldots,p$, and

$$\underset{\sim}{U}_1\underset{\sim}{A} \overset{d}{=} \underset{\sim}{Y}\underset{\sim}{R}^{-1} = (\underset{\sim}{y}_1,\ldots,\underset{\sim}{y}_p)\underset{\sim}{R}^{-1} = \underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{R}^{-1}$$

where $\underset{\sim}{Q}\underset{\sim}{T}$ is the Bartlett decomposition for $(\underset{\sim}{y}_1,\ldots,\underset{\sim}{y}_p)$ and $\underset{\sim}{Q}'\underset{\sim}{Q} = \underset{\sim}{I}_p$. By Lemma 4, we have $\underset{\sim}{T}\underset{\sim}{R}^{-1} \overset{d}{=} \underset{\sim}{A}$ which completes the proof. Q.E.D.

<u>Remark</u>. Let $\underset{\sim}{A} = (a_{ij})$, $\underset{\sim}{a}_i = (a_{1i},\ldots,a_{ii})'$, $\underset{\sim}{a}_i^* = (a_{1i},\ldots,a_{i-1,i})'$ and $\underset{\sim}{a}_i^{(2)} = (a_{1i}^2,\ldots,a_{ii}^2)$, $i = 1,\ldots,p$. Then by Theorem 2, we obtain the following facts:

(1) $\underset{\sim}{a}_1,\ldots,\underset{\sim}{a}_p$ are independent;

(2) $\underset{\sim}{a}_k^* \overset{d}{=} \underset{\sim}{u}_k$, where $\underset{\sim}{u}_k$ is the first (k-1)-component subvector of $\underset{\sim}{u}^{(n)}$, $k = 2,\ldots,p$; and

(3) $\underset{\sim}{a}_k^{(2)} \sim D_k(\frac{1}{2},\ldots,\frac{1}{2},\frac{1}{2}(n-k))$, $k = 2,\ldots,p$.

<u>Corollary 1</u>. $\underset{\sim}{X} \in F_2$ iff $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{U}_1\underset{\sim}{A}\underset{\sim}{R}$, where $\underset{\sim}{U}_1$, $\underset{\sim}{A}$, and $\underset{\sim}{R} = \mathrm{diag}(R_1,\ldots,R_p) \ge 0$ are independent, and $\underset{\sim}{A}$ is given by Theorem 2.

<u>Theorem 3</u>. Let $\underset{\sim}{U}_3 \overset{d}{=} \underset{\sim}{U}_1\underset{\sim}{B}$; then $\underset{\sim}{B} \overset{d}{=} \underset{\sim}{T}/(\mathrm{tr}\underset{\sim}{T}'\underset{\sim}{T})^{1/2}$.

Proof. By using the above notation, we have $\underset{\sim}{Y}'(\text{tr}\underset{\sim}{Y}'\underset{\sim}{Y})^{1/2} \overset{d}{=} \underset{\sim}{U}_3$ and $\underset{\sim}{T}'\underset{\sim}{T}/\text{tr}\underset{\sim}{T}'\underset{\sim}{T} = \underset{\sim}{Y}'\underset{\sim}{Y}/\text{tr}\underset{\sim}{Y}'\underset{\sim}{Y} \overset{d}{=} \underset{\sim}{B}'\underset{\sim}{B}$. Note $\underset{\sim}{B}$ is also an upper triangular matrix with positive diagonal elements, then $\underset{\sim}{B} \overset{d}{=} \underset{\sim}{T}/(\text{tr}\underset{\sim}{T}'\underset{\sim}{T})^{1/2}$. Q.E.D.

Corollary 1. $\underset{\sim}{X} \in \mathcal{F}_3$ iff $\underset{\sim}{X} \overset{d}{=} R\underset{\sim}{U}_1\underset{\sim}{B}$, where $R \geq 0$, $\underset{\sim}{U}_1$, and $\underset{\sim}{B}$ are independent, and $\underset{\sim}{B}$ is given by Theorem 3.

3.2 Classes of marginal distributions.

Let $\mathcal{F}_i^c$ $(i=1,2,3)$ denote the set of first columns of $\underset{\sim}{X}$'s in $\mathcal{F}_i$ $(i=1,2,3)$, i.e., $\underset{\sim}{x} \in \mathcal{F}_i^c$ iff there exist $\underset{\sim}{x}_2,\ldots,\underset{\sim}{x}_p$ such that $\underset{\sim}{X} = (\underset{\sim}{x},\underset{\sim}{x}_2,\ldots,\underset{\sim}{x}_p) \in \mathcal{F}_i$ $(i=1,2,3)$. Similarly, $\mathcal{F}_i^r$ indicates a set of the first row vector of $\underset{\sim}{X}$ in $\mathcal{F}_i$ $(i=1,2,3)$. Clearly $\mathcal{F}_1^c \supset \mathcal{F}_3^c \supset \mathcal{F}_3^c$ and $\mathcal{F}_1^r \supset \mathcal{F}_2^r \supset \mathcal{F}_3^r$, since $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \mathcal{F}_3$. Also $\mathcal{F}_2^c = \mathcal{F}_1^c$, in fact if $\underset{\sim}{x} \in \mathcal{F}_1^c$ let $\underset{\sim}{x}_2,\ldots,\underset{\sim}{x}_p$ be $p-1$ $n\times 1$ random vectors such that $\underset{\sim}{x},\underset{\sim}{x}_2,\ldots,\underset{\sim}{x}_p$ are iid; thus $\underset{\sim}{X} = (\underset{\sim}{x},\underset{\sim}{x}_2,\ldots,\underset{\sim}{x}_p) \in \mathcal{F}_2$. Here we use a useful fact that if $\underset{\sim}{X} = (\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_p)$ with $\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_p$ i.i.d. and $\underset{\sim}{x}_1 \sim EC_n(\underset{\sim}{0},\underset{\sim}{I}_n,\phi)$, then $\underset{\sim}{X} \in \mathcal{F}_2$. Naturally, one may ask if $\mathcal{F}_3^c = \mathcal{F}_2^c$ holds. But that is not true. First, we need the following lemma.

Lemma 7. $\Omega_n(\underset{\sim}{t}'\underset{\sim}{t})$ with $\underset{\sim}{t} \in R^{n+1}$ is not an n+1-dimensional c.f.

Proof. Suppose $\Omega_n(\underset{\sim}{t}'\underset{\sim}{t})$ with $\underset{\sim}{t} \in R^{n+1}$ is an (n+1)-dimensional c.f. Then there exists a distribution function F such that (c.f. Cambanis, Huang and Simons (1981))

$$\Omega_n(u) = \int_0^\infty \Omega_{n+1}(ur^2)\, dF(r), \qquad u \geq 0 \,, \tag{3.1}$$

i.e., $\underset{\sim}{u}^{(n)} \overset{d}{=} R\underset{\sim}{u}_n$, where R is independent of $\underset{\sim}{u}_n$, $R \geq 0$, and $\underset{\sim}{u}_n$ is the subvector of $\underset{\sim}{u}^{(n+1)}$ with the first n components. Since $\underset{\sim}{u}_n$ has a pdf and $P(R=0) = P(R\underset{\sim}{u}_n = 0) = P(\underset{\sim}{u}^{(n)} = \underset{\sim}{0}) = 0$, $\underset{\sim}{u}^{(n)}$ has a pdf. This is a contradiction, which completes the proof. Q.E.D.

Theorem 4. The set $\mathcal{F}_3^c$ is a proper subset of $\mathcal{F}_2^c$ if $p > 1$.

Proof. Let $\underset{\sim}{u} \overset{d}{=} \underset{\sim}{u}^{(n)}$. Clearly, $\underset{\sim}{u} \in \mathcal{F}_2^c$, we want to point out $\underset{\sim}{u} \notin \mathcal{F}_3^c$. Suppose $\underset{\sim}{u} \in \mathcal{F}_3^c$; then there exist $\underset{\sim}{u}_2, \ldots, \underset{\sim}{u}_p$ such that $\underset{\sim}{U} = (\underset{\sim}{u}, \underset{\sim}{u}_2, \ldots, \underset{\sim}{u}_p) \in \mathcal{F}_3$. Let $\phi(\mathrm{tr}\underset{\sim}{T}'\underset{\sim}{T})$ denote the c.f. of $\underset{\sim}{U}$. Then the c.f. of $\underset{\sim}{u}$ is $\phi(\underset{\sim}{t}_1'\underset{\sim}{t}_1) = \Omega_n(\underset{\sim}{t}_1'\underset{\sim}{t}_1)$, $\underset{\sim}{t}_1 \in R^n$, i.e., $\phi(\cdot) = \Omega_n(\cdot)$. That means that $\Omega_n(\underset{\sim}{t}'\underset{\sim}{t}) = \phi(\underset{\sim}{t}'\underset{\sim}{t})$, $\underset{\sim}{t} \in R^{np}$, is a c.f. By Lemma 7, the contradiction proves the theorem. Q.E.D.

Let us consider the row marginal distributions. First, we want to point out that $\mathcal{F}_2^r$ is a proper subset of $\mathcal{F}_1^r$.

Lemma 8. Suppose $\underset{\sim}{X} = (\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_p) = (\underset{\sim}{x}_{(1)}, \ldots, \underset{\sim}{x}_{(n)})' \in \mathcal{F}_2$ and the covariance of $\underset{\sim}{x}_{(1)}$ exists. Then

(1) $\mathrm{Cov}(\underset{\sim}{x}_{(i)}, \underset{\sim}{x}_{(j)}) = \delta_{ij}\underset{\sim}{\Lambda}_i$, where $\underset{\sim}{\Lambda}_i$ is a diagonal matrix and $\delta_{ii} = 1$, and $\delta_{ij} = 0$, $i \neq j$, $i,j = 1, \ldots, n$, and

(2) $\mathrm{Cov}(\underset{\sim}{x}_i, \underset{\sim}{x}_j) = \delta_{ij}\sigma_{ii}^2\underset{\sim}{I}_n$, where σ_{ii}^2 will be given in the proof, $i,j = 1, \ldots, p$.

Proof. Clearly, $\underset{\sim}{x}_{(1)}, \ldots, \underset{\sim}{x}_{(n)}$ are identically distributed and

$$\underset{\sim}{X} \overset{d}{=} \underset{\sim}{U}_2\underset{\sim}{R} = (R_1\underset{\sim}{u}_1, \ldots, R_p\underset{\sim}{u}_p) , \tag{3.2}$$

where $\underset{\sim}{R}, \underset{\sim}{u}_1,\ldots,\underset{\sim}{u}_p$ are independent. As $E\underset{\sim}{U}_2 = \underset{\sim}{0}$, we have $E\underset{\sim}{x}_{(k)} = \underset{\sim}{0}$, and $E\underset{\sim}{x}_j = \underset{\sim}{0}$ for $k=1,\ldots,n;\ j=1,\ldots,p$. By (3.2)

$$\mathrm{Cov}(\underset{\sim}{x}_{(i)},\underset{\sim}{x}_{(j)}) = E\underset{\sim}{x}_{(i)}\underset{\sim}{x}'_{(j)} = \mathrm{diag}(ER_1^2 Eu_{i1}u_{j1},\ldots,ER_p^2 Eu_{ip}u_{jp}) .$$

The first assertion follows from $Eu_{ik}u_{jk} = 0$ for $i \neq j;\ k = 1,\ldots,p$. Similarly, $E\underset{\sim}{x}_i\underset{\sim}{x}'_j = ER_iR_jE\underset{\sim}{u}_i\underset{\sim}{u}'_j = \delta_{ij}ER_i^2\underset{\sim}{I}_n/n$. and the Lemma follows.

Let $\underset{\sim}{x}_{(1)},\ldots,\underset{\sim}{x}_{(n)}$ be i.i.d., $\underset{\sim}{x}_{(1)} \sim N(\underset{\sim}{0},\underset{\sim}{\Sigma})$ and $\underset{\sim}{\Sigma}$ is not a diagonal matrix; by Lemma 8, then $\underset{\sim}{x}_{(1)} \notin F_2^r$, but $\underset{\sim}{x}_{(1)} \in F_1^r$. Thus F_2^r is a proper subset of F_1^r.

Theorem 5. Suppose $\underset{\sim}{X} = (\underset{\sim}{x}_{(1)},\ldots,\underset{\sim}{x}_{(n)})' \in F_2$, then $\underset{\sim}{X} \in F_3$ iff $\underset{\sim}{x}_{(1)} \in F_3^r$.

Proof. The "only if" part is obvious. Suppose $\underset{\sim}{x}_{(1)} \in F_3^r$; then $\underset{\sim}{x}_{(1)}$ has a c.f. $\phi(\underset{\sim}{t}'_{(1)}\underset{\sim}{t}_{(1)})$, where $\phi(\underset{\sim}{t}'_{(1)}\underset{\sim}{t}_{(1)}+\cdots+\underset{\sim}{t}'_{(n)}\underset{\sim}{t}_{(n)})$ is a c.f. in R^{np}. On the other hand, since $\underset{\sim}{X} \in F_2$, $\underset{\sim}{X}$ has a c.f. $\psi(\underset{\sim}{t}'_1\underset{\sim}{t}_1,\ldots,\underset{\sim}{t}'_p\underset{\sim}{t}_p)$. Hence we must have

$$\phi(r_1^2+\cdots+r_p^2) = \psi(r_1^2,\ldots,r_p^2), \quad \text{for } r_i \geq 0,\ i = 1,\ldots,p, \tag{3.3}$$

because they are all the c.f. of $\underset{\sim}{x}_{(1)}$. By (3.3), we have $\phi(\underset{\sim}{t}'_1\underset{\sim}{t}_1+\cdots+\underset{\sim}{t}'_p\underset{\sim}{t}_p) = \psi(\underset{\sim}{t}'_1\underset{\sim}{t}_1,\ldots,\underset{\sim}{t}'_p\underset{\sim}{t}_p)$ for all $\underset{\sim}{t}_i \in R^n$, $i=1,\ldots,p$, i.e., $\underset{\sim}{x} \in F_3$. The theorem follows. Q.E.D.

Corollary 1. Suppose $\underset{\sim}{X} \in F_2$. Then $\underset{\sim}{x}_{(1)} \in F_3^r$ iff $\underset{\sim}{x}_{(1)} \sim S_p(\phi)$.

Proof. The "only if" part is trivial. Now suppose $\underset{\sim}{x}_{(1)} \sim S_p(\phi)$. By Theorem 5, we get $X \in F_3$ since $X \in F_2$ and $\underset{\sim}{x}_{(1)} \in F_3^r$. Q.E.D.

Corollary 2. The first row $\underset{\sim}{u}_{(1)}$ of $\underset{\sim}{U}_1$ is not in $\mathcal{F}_2^r$.

Proof. Assume $\underset{\sim}{u}_{(1)} \varepsilon \mathcal{F}_2^r$, then there exists $\underset{\sim}{Y}$ such that $\underset{\sim}{X} = (\underset{\sim}{u}_{(1)}, \underset{\sim}{Y})' \varepsilon \mathcal{F}_2$. As $\underset{\sim}{U}_1$ is right spherical (Dawid (1977)), therefore $\underset{\sim}{u}_{(1)} \sim S_p(\phi)$ and $\underset{\sim}{u}_{(1)} \varepsilon \mathcal{F}_3^r$ by $\underset{\sim}{X} \varepsilon \mathcal{F}_2$ and Corollary 1 of Theorem 5. However, it is impossible (cf. the following example). Hence, $\underset{\sim}{u}_{(1)} \notin \mathcal{F}_2^r$. Q.E.D.

Example 2. Suppose $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{U}_1\underset{\sim}{A}$, where $\underset{\sim}{U}_1$ and $\underset{\sim}{A}$ are independent, and $\underset{\sim}{A} = \mathrm{diag}(a_1,\ldots,a_p)$, $0 < p_i = P(a_i=1) = 1-P(a_i=0) < 1$, $i=1,\ldots,p$, $a_1+\cdots+a_p = 1$. Clearly $\underset{\sim}{X} \varepsilon \mathcal{F}_1$, but $\underset{\sim}{X} \notin \mathcal{F}_2$ by Theorem 2. We, however, want to show $\underset{\sim}{x}_{(1)} \overset{d}{=} \underset{\sim}{A}\underset{\sim}{u}_{(1)} \varepsilon \mathcal{F}_2^r$, where $\underset{\sim}{u}_{(1)}$ is the first row of $\underset{\sim}{U}_1$. It can be shown that the c.f. of $\underset{\sim}{u}_{(1)}$ is $\Omega_n(t_1^2+\cdots+t_p^2)$ by the sphericity of $\underset{\sim}{U}_1$. And $\underset{\sim}{x}_{(1)}$ has a c.f.

$$(3.4) \qquad \psi(t_1^2,\ldots,t_p^2) = \int \Omega_n(a_1^2t_1^2+\cdots+a_p^2t_p^2)\, dF(a_1,\ldots,a_p)$$

$$= \sum_{i=1}^{p} p_i\Omega_n(t_i^2) \ .$$

By (3.4), we have

$$\psi(\underset{\sim}{t}_1'\underset{\sim}{t}_1,\ldots,\underset{\sim}{t}_p'\underset{\sim}{t}_p) = \sum_{i=1}^{p} p_i\Omega_n(\underset{\sim}{t}_i'\underset{\sim}{t}_i), \quad \underset{\sim}{t}_i \varepsilon R^n, \ i = 1,\ldots,p \ .$$

As $\Omega_n(\underset{\sim}{t}_i'\underset{\sim}{t}_i)$ is a c.f. in R^n and $\Sigma_{i=1}^p p_i = 1$, $p_i > 0$, $i = 1,\ldots,p$, hence $\psi(\underset{\sim}{t}_1'\underset{\sim}{t}_1,\ldots,\underset{\sim}{t}_p'\underset{\sim}{t}_p)$ is the c.f. of some $\underset{\sim}{Y}$ in $\mathcal{F}_2$ and $\underset{\sim}{y}_{(1)} \overset{d}{=} \underset{\sim}{x}_{(1)}$, where $\underset{\sim}{y}_{(1)}$ is the first row of $\underset{\sim}{Y}$; that means $\underset{\sim}{x}_{(1)} \varepsilon \mathcal{F}_2^r$.

Example 2 shows us that $\mathcal{F}_2$, related to $\mathcal{F}_1$, cannot be characterized by its row marginal distributions. But for $\mathcal{F}_3$, related to $\mathcal{F}_2$, it can by Theorem 5. Further, it is easy to show that if $\underset{\sim}{X}, \underset{\sim}{Y} \in \mathcal{F}_2$ and $\underset{\sim}{x}_{(1)} \overset{d}{=} \underset{\sim}{y}_{(1)}$, we have $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{Y}$. However, there is no such property for $\mathcal{F}_1$.

3.3 Marginal densities.

Let $\underset{\sim}{X} \in \mathcal{F}_i$, $i = 1,2,3$. In general it is not necessary that $\underset{\sim}{X}$ has a density. If $\underset{\sim}{X}$ satisfies some suitable additional condition on $\underset{\sim}{X}$, it will have marginal densities.

Suppose $\underset{\sim}{X} \overset{d}{=} R\underset{\sim}{U}_3 \in \mathcal{F}_3$; if $P(\underset{\sim}{X}=\underset{\sim}{0}) = P(R=0) = 0$, then all marginal densities exist (Kelker (1970)). Suppose $\underset{\sim}{X} \overset{d}{=} (R_1\underset{\sim}{u}_1,\ldots,R_p\underset{\sim}{u}_p) \in \mathcal{F}_2$; if $P(\underset{\sim}{x}_i=\underset{\sim}{0}) = P(R_i=0) = 0$, $\underset{\sim}{x}_i$ has all marginal densities; if $P(\underset{\sim}{x}_i=\underset{\sim}{0}) = 0$, $i=1,\ldots,p$, then $\underset{\sim}{X}$ has marginal densities of a set of elements such that at least one element in each colum of $\underset{\sim}{X}$ has been deleted. Also, we can prove that if $\underset{\sim}{X} \in \mathcal{F}_1$ and $P(|\underset{\sim}{X}'\underset{\sim}{X}|=0) = 0$, then $(x_{11},\ldots,x_{n-1,1},x_{12},\ldots,x_{n-2,2},\ldots,x_{1p},\ldots,x_{n-p,p})$ and all its subsets have marginal densities.

Further, if $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{U}_2\underset{\sim}{R} \in \mathcal{F}_2$, it is easy to see that $\underset{\sim}{X}$ has a pdf $f_{\underset{\sim}{X}}(\underset{\sim}{x}_1'\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_p'\underset{\sim}{x}_p)$ iff $\underset{\sim}{R}$ has a pdf $f_{\underset{\sim}{R}}(r_1,\ldots,r_p)$, and there exists the following relationship between them (cf. Zhang and Fang (1982), Ch. 9):

$$f_{\underset{\sim}{R}}(r_1,\ldots,r_p) = (2^p\pi^{np/2}(\Gamma(\tfrac{1}{2}n))^{-p})(r_1\cdots r_p)^{n-1}f_X(r_1^2,\ldots,r_p^2),\ r_1,\ldots,r_p > 0.$$

Similarly, suppose $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{U}_1\underset{\sim}{A} \in \mathcal{F}_1$, where $\underset{\sim}{A} = (a_{ij})$ is an upper triangular matrix with positive diagonal elements; then $\underset{\sim}{X}$ has a pdf $f_{\underset{\sim}{X}}(\underset{\sim}{X}'\underset{\sim}{X})$

iff $\underset{\sim}{A}$ has a pdf $f_{\underset{\sim}{A}}(\underset{\sim}{A})$, and $f_{\underset{\sim}{A}}$ is related to $f_{\underset{\sim}{X}}$ as follows:

$$f_{\underset{\sim}{A}}(\underset{\sim}{A}) = 2^p \pi^{pn/2-p(p-1)/4} \prod_{i=1}^{p} a_{ii}^{n-i} f_{\underset{\sim}{X}}(\underset{\sim}{A}'\underset{\sim}{A}) \Big/ \prod_{i=1}^{p} \Gamma(\tfrac{1}{2}(n-i+1)).$$

(cf. Srivastava and Khatri (1979)).

4. The Class of Spherical Matrix Distributions.

Let $\underset{\sim}{A} \geq 0$ be a $p \times p$ matrix and $\lambda_1 \geq \cdots \geq \lambda_p \geq 0$ be the eigenvalues of $\underset{\sim}{A}$. We write $\lambda(\underset{\sim}{A}) = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$.

Definition 2. (Dawid (1977)). A random $n \times p$ matrix $\underset{\sim}{X}$ is called spherical if $\underset{\sim}{X}$ and $\underset{\sim}{X}'$ are left-spherical, i.e., $\underset{\sim}{P}\underset{\sim}{X}\underset{\sim}{Q} \overset{d}{=} \underset{\sim}{X}$ for each $\underset{\sim}{P} \in O(n)$ and $\underset{\sim}{Q} \in O(p)$. Denote $F_s = \{\underset{\sim}{X}: \underset{\sim}{X}$ is spherical$\}$.

Lemma 9. (Dawid). $\underset{\sim}{X} \in F_s$ iff $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{U}_1\underset{\sim}{\Lambda}\underset{\sim}{V}$, where $\underset{\sim}{U}_1, \underset{\sim}{\Lambda}$, and $\underset{\sim}{V}$ are independent, $\underset{\sim}{\Lambda} = \lambda((\underset{\sim}{X}'\underset{\sim}{X})^{1/2})$, $\underset{\sim}{\Gamma}\underset{\sim}{V} \overset{d}{=} \underset{\sim}{V}$, $\underset{\sim}{V}'\underset{\sim}{V} = \underset{\sim}{I}_p$ for each $\underset{\sim}{\Gamma} \in O(p)$.

The class of F_s was studied by Dawid. The c.f. of $\underset{\sim}{X}$ in F_s must have the form $\phi(\lambda(\underset{\sim}{T}'\underset{\sim}{T}))$, because the maximum invariant of $\underset{\sim}{T}(\underset{\sim}{T}: n \times p)$ under the transformation $\underset{\sim}{P}\underset{\sim}{T}\underset{\sim}{Q}$ for each $\underset{\sim}{P} \in O(n)$ and each $\underset{\sim}{Q} \in O(p)$, is $\lambda(\underset{\sim}{T}'\underset{\sim}{T})$. It is easy to see $F_1 \supset F_s \supset F_3$ and $\underset{\sim}{U}_1 \in F_s$. If $\underset{\sim}{X} \in F_2$, it is not necessary that $\underset{\sim}{X} \in F_s$, and vice versa. For example, the $\underset{\sim}{X}$ in Example 1 belongs to F_s, but $\underset{\sim}{X} \notin F_2$. By the following theorem, we see that $\underset{\sim}{U}_2 \notin F_s$.

Theorem 6. $F_3 = F_s \cap F_2$.

Proof. Clearly, $F_3 \subset F_s \cap F_2$. Conversely, if $\underset{\sim}{X} \in F_s \cap F_2$, the

fact $\underset{\sim}{X} = (\underset{\sim}{x}_{(1)},\ldots,\underset{\sim}{x}_{(n)})' \in F_s$ implies $\underset{\sim}{x}_{(1)} \sim S_p(\phi)$ and $\underset{\sim}{X} \in F_3$ from Corollary 1 of Theorem 5. The theorem follows. Q.E.D.

<u>Theorem 7</u>. Let $\underset{\sim}{U}_3 \overset{d}{=} \underset{\sim}{U}_1\underset{\sim}{\Lambda}\underset{\sim}{V}$, where $\underset{\sim}{U}_1, \underset{\sim}{\Lambda}$, and $\underset{\sim}{V}$ have the meaning in Lemma 9. Denote $\underset{\sim}{\Lambda}^2 = \mathrm{diag}(\lambda_1,\ldots,\lambda_p)$, $\lambda_1 \geq \cdots \geq \lambda_p \geq 0$, then

$$(\lambda_1,\ldots,\lambda_p) \overset{d}{=} (w_1,\ldots,w_p)/(w_1+\cdots+w_p),$$

where $w_1,\ldots,w_p$ are p eigenvalues of $\underset{\sim}{W} \sim W_p(n,\underset{\sim}{I}_p)$, and $(\lambda_1,\ldots,\lambda_{p-1})$ has a joint density

$$f(\lambda_1,\ldots,\lambda_{p-1}) = \frac{\pi^{p/2}\Gamma(np/2)}{\prod_{\alpha=0}^{p-1}\Gamma(\frac{p-\alpha}{2})\Gamma(\frac{n-\alpha}{2})}|\lambda_1,\ldots,\lambda_p|^{\frac{1}{2}(n-p-1)} \prod_{1\leq i<j\leq p}(\lambda_i-\lambda_j) \tag{3.5}$$

$$(\lambda_1 > \cdots > \lambda_{p-1} > 0 \text{ and } \lambda_p = 1-\lambda_1-\cdots-\lambda_{p-1} > 0)\ .$$

and $(\lambda_1,\ldots,\lambda_{p-1})$ is independent of $w = w_1+\cdots+w_p$.

<u>Proof</u>. Let $\underset{\sim}{Y} = (\underset{\sim}{y}_1,\ldots,\underset{\sim}{y}_p)$ with $\underset{\sim}{y}_1,\ldots,\underset{\sim}{y}_p$ i.i.d. and $\underset{\sim}{y}_1 \sim N_n(\underset{\sim}{0},\underset{\sim}{I}_n)$. Then $\underset{\sim}{Y}/(\mathrm{tr}\underset{\sim}{Y}'\underset{\sim}{Y})^{1/2} \overset{d}{=} \underset{\sim}{U}_3 \overset{d}{=} \underset{\sim}{U}_1\underset{\sim}{\Lambda}\underset{\sim}{V}$ and $\lambda(\underset{\sim}{Y}'\underset{\sim}{Y}/\mathrm{tr}\underset{\sim}{Y}'\underset{\sim}{Y}) \overset{d}{=} \mathrm{diag}(\lambda_1,\ldots,\lambda_p)$. Note that $\lambda(\underset{\sim}{Y}'\underset{\sim}{Y}/\mathrm{tr}\underset{\sim}{Y}'\underset{\sim}{Y}) = \lambda(\underset{\sim}{Y}'\underset{\sim}{Y})/\mathrm{tr}(\underset{\sim}{Y}'\underset{\sim}{Y})$ and $\underset{\sim}{Y}'\underset{\sim}{Y} \equiv \underset{\sim}{W} \sim W_p(n,\underset{\sim}{I}_p)$, $\mathrm{tr}(\underset{\sim}{Y}'\underset{\sim}{Y}) = w_1+\cdots+w_p$ and $\lambda(\underset{\sim}{W}) = \mathrm{diag}(w_1,\ldots,w_p)$, the first part of the theorem follows. To check (3.5), we note that $(w_1,\ldots,w_p)$ has the following pdf (cf. Anderson (1958) or Zhang and Fang (1982)):

$$(3.6)\qquad \frac{\pi^{p/2}}{2^{np/2}\prod_{\alpha=0}^{p-1}\Gamma(\frac{p-\alpha}{2})\Gamma(\frac{n-\alpha}{2})}\prod_{i=1}^{p} w_i^{1/2(n-p-1)}\prod_{i<j}(w_i-w_j)e^{-\frac{1}{2}(w_1+\cdots+w_p)}$$

$$w_1 > \cdots > w_p > 0 .$$

Taking the transformation

$$\lambda_i = w_i/(w_1+\cdots w_p),\quad i = 1,\ldots,p-1 ,$$

$$\lambda = w_1+\cdots+w_p ,$$

the Jacobian is λ^{p-1}. Let $\lambda_p = 1-\lambda_1 -\cdots- \lambda_{p-1}$. Now (3.6) becomes

$$\left[\frac{\pi^{p/2}\,\Gamma(np/2)}{\prod_{\alpha=0}^{p-1}\Gamma(\frac{p-\alpha}{2})\Gamma(\frac{n-\alpha}{2})}\prod_{i=1}^{p}\lambda_i^{\frac{1}{2}(n-p-1)}\prod_{1\le i<j\le p}(\lambda_i-\lambda_j)\right]\left[\frac{1}{2^{np/2}\Gamma(\frac{np}{2})}\lambda^{np/2-1}e^{-\lambda/2}\right]$$

which completes the proof. Q.E.D.

Assume $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{U}_1\underset{\sim}{\Lambda}\underset{\sim}{V} \in F_s$ (cf. Lemma 9), then $\underset{\sim}{X}$ has a pdf $f_{\underset{\sim}{X}}(\lambda(\underset{\sim}{X}'\underset{\sim}{X}))$ if $\underset{\sim}{\Lambda}$ has a pdf $f_{\underset{\sim}{\Lambda}}(\lambda_1,\ldots,\lambda_p)$, and $f_{\underset{\sim}{\Lambda}}$ is related to $f_{\underset{\sim}{X}}$ as follows (cf. Theorem 13.3.1. of Anderson (1958))

$$f_{\underset{\sim}{\Lambda}}(\lambda_1,\ldots,\lambda_p) = \frac{\pi^{p(n+1)/2}}{\prod_{\alpha=0}^{p-1}\Gamma(\frac{p-\alpha}{2})\Gamma(\frac{n-\alpha}{2})}(\lambda_1\ \cdots\ \lambda_p)^{\frac{1}{2}(n-p-1)}$$

$$\prod_{i<j}(\lambda_i-\lambda_j)\ f_{\underset{\sim}{X}}(\mathrm{diag}(\lambda_1,\ldots,\lambda_p)) ,\quad \lambda_1 > \cdots > \lambda_p > 0 .$$

Theorem 8. Assume $\underset{\sim}{X} \in F_s$ with independent columns (or rows), then $\underset{\sim}{X}$ must be normal.

Proof. From the assumption, the c.f. of $\underset{\sim}{X}$ is $\Pi_1^p \phi(\underset{\sim}{t}_i'\underset{\sim}{t}_i)$, i.e., $\underset{\sim}{X} \in F_2$. By Theorem 6, $\underset{\sim}{X} \in F_3$. The assertion follows from Kelker (1970). Q.E.D.

This theorem shows that we, in general, should consider dependent sample theory in F_s. If $\underset{\sim}{X} \in F_3$, the c.f. of $\underset{\sim}{X}$ has the form $\phi(\lambda(\underset{\sim}{T}'\underset{\sim}{T})) = \phi(\mathrm{diag}(\lambda_1,\ldots,\lambda_p)) = \psi(\lambda_1+\cdots+\lambda_p)$, i.e., the c.f. is the function of $\lambda_1 +\ldots+ \lambda_p$. Here $\lambda_1,\ldots,\lambda_p$ are the eigenvalues of $\underset{\sim}{T}'\underset{\sim}{T}$ in $Ee^{\mathrm{tr}\, \underset{\sim}{T}'\underset{\sim}{X}}$. We may consider other functions of $\lambda_1,\ldots,\lambda_p$ to obtain other different subclasses of F_s.

5. Applications.

Let $F_3^+ = \{\underset{\sim}{X} \in F_3\colon P(\underset{\sim}{X}=\underset{\sim}{0})=0\}$, $F_2^+ = \{\underset{\sim}{X} \in F_2\colon P(\underset{\sim}{x}_i=\underset{\sim}{0})=0,\ i=1,\ldots,p\}$, and $F_1^+ = \{\underset{\sim}{X} \in F_1\colon P(|\underset{\sim}{X}'\underset{\sim}{X}|=0)=0\}$. We call a statistic $t(\underset{\sim}{X})$ distribution free on F_i^+ if $t(\underset{\sim}{X}) \overset{d}{=} t(\underset{\sim}{Y})$ for any $\underset{\sim}{X},\underset{\sim}{Y} \in F_i^+$; $i = 1,2,3$, respectively.

Theorem 9. Suppose $t(\underset{\sim}{X})$ is a statistic. Then

(a) $t(\underset{\sim}{X})$ is distribution free on F_3^+ iff $t(a\underset{\sim}{X}) \overset{d}{=} t(\underset{\sim}{X})$ for each $a > 0$;

(b) $t(\underset{\sim}{X})$ is distribution free on F_2^+ iff $t(\underset{\sim}{X}\underset{\sim}{r}) \overset{d}{=} t(\underset{\sim}{X})$ for each $\underset{\sim}{r} = \mathrm{diag}(r_1,\ldots,r_p)$, $r_i > 0$, $i=1,\ldots,p$; and

(c) $t(\underset{\sim}{X})$ is distribution free on F_1^+ iff $t(\underset{\sim}{X}\underset{\sim}{A}) \overset{d}{=} t(\underset{\sim}{X})$ for each $\underset{\sim}{A}$, an upper triangular matrix with positive diagonal elements.

Proof. We only prove (b); the other proofs are similar. The "only if" part is trivial. Suppose $\underset{\sim}{X} \in F_2^+$ and $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{U}_2\underset{\sim}{R}$, where $\underset{\sim}{R} = \mathrm{diag}(R_1,\ldots,R_p)$, $R_i > 0$, $i = 1,\ldots,p$, and $\underset{\sim}{U}_2$ is independent of $\underset{\sim}{R}$. Then for each Borel function $h \geq 0$, we have (by using the assumption that $t(\underset{\sim}{X}\underset{\sim}{r}) \overset{d}{=} t(\underset{\sim}{X})$ for each $\underset{\sim}{r}$)

$$E(h(t(\underset{\sim}{X}))) = E(h(t(\underset{\sim}{U}_2\underset{\sim}{R}))) = E_{\underset{\sim}{R}}(E(h(t(\underset{\sim}{U}_2\underset{\sim}{R}))|\underset{\sim}{R}))$$

$$= E_{\underset{\sim}{R}}(E_{\underset{\sim}{U}_2}(h(t(\underset{\sim}{U}_2\underset{\sim}{R})))) = E_{\underset{\sim}{R}}(E_{\underset{\sim}{U}_2}(h(t(\underset{\sim}{U}_2))))$$

$$= E(h(t(\underset{\sim}{U}_2))) ,$$

which is independent of $\underset{\sim}{X}$ in F_2^+, the sufficiency follows. Q.E.D.

Remark 1. The assertions (a) and (c) are essentially from Kariya (1981a), but the statement here is a little different from his.

Remark 2. In this paper we have only studied F_1, F_2, and F_3 for the case of central standard spherical matrices. If we consider the following transformations: $F_1 \to \{\underset{\sim}{X}+\underset{\sim}{M}: \underset{\sim}{X} \varepsilon F_1\}$, $F_2 \to \{\underset{\sim}{X}\underset{\sim}{\Sigma}^{1/2}+\underset{\sim}{M}: \underset{\sim}{X} \varepsilon F_2\}$ and $F_3 \to \{\underset{\sim}{X}\underset{\sim}{\Sigma}^{1/2}+\underset{\sim}{M}: \underset{\sim}{X} \varepsilon F_3\}$, where $\underset{\sim}{M}$ is an $n \times p$ constant matrix and $\underset{\sim}{\Sigma} = \underset{\sim}{\Sigma}^{1/2}\underset{\sim}{\Sigma}^{1/2}$ is a positive definite matrix, we can generalize our results.

In the rest of this section denote $\underset{\sim}{X} \varepsilon F_i^+$ $(i = 1,2 \text{ or } 3)$ and $\underset{\sim}{W} = \underset{\sim}{X}'\underset{\sim}{D}_n\underset{\sim}{X}$, where $\underset{\sim}{D}_n = \underset{\sim}{I}_n - \underset{\sim}{1}_n\underset{\sim}{1}_n'/n$ and $\underset{\sim}{1}_n = (1,\ldots,1)'$. For convenience of applications, some basic invariant statistics in F_1^+, in F_2^+, and in F_3^+ are listed in Table 1. They are

(1) The Wilks statistic. Let $\underset{\sim}{X} = (\underset{\sim}{X}_1',\underset{\sim}{X}_2')'$, $\underset{\sim}{X}_1$: $n_1 \times p$, $\underset{\sim}{X}_2$: $n_2 \times p$, $n_1 \geq p$, and $n_2 \geq p$. Let $\underset{\sim}{W}_k = \underset{\sim}{X}_k'\underset{\sim}{D}_{n_k}\underset{\sim}{X}_k$, $k = 1,2$. The Wilks statistic is

$$t_1(\underset{\sim}{X}) = |\underset{\sim}{W}_1|/|\underset{\sim}{W}_1+\underset{\sim}{W}_2| .$$

(2) The multivariate Beta statistic. Let $\underset{\sim}{W}_0 = \underset{\sim}{W}_1+\underset{\sim}{W}_2$, where $\underset{\sim}{W}_1$ and $\underset{\sim}{W}_2$ are given in (1). The multivariate Beta statistic is

$$t_2(\underset{\sim}{X}) = \underset{\sim}{W}_0^{-1/2}\underset{\sim}{W}_1\underset{\sim}{W}_0^{-1/2} .$$

(3) The Hotelling T^2-statistic. It is

$$t_3(\underset{\sim}{X}) = n(n-1)\bar{\underset{\sim}{x}}'\underset{\sim}{W}^{-1}\bar{\underset{\sim}{x}} ,$$

where $\bar{\underset{\sim}{x}} = \underset{\sim}{X}'\underset{\sim}{1}_n/n.$

Table 1.

Distribution-free Properties of the Invariant Statistics

Statistics	$\mathcal{F}_1^+$	$\mathcal{F}_2^+$	$\mathcal{F}_3^+$
$t_1(X)$	free	free	free
$t_2(X)$	free	free	free
$t_3(X)$	free	free	free
$t_4(X)$	free	free	free
$t_5(X)$	not	free	free
$t_6(X)$	not	free	free
$t_7(X)$	not	not	free
$t_8(X)$	not	not	not
$t_9(X)$	not	not	not

(4) The statistic testing equality of covariance matrices. Partition $\underset{\sim}{X} = (\underset{\sim}{X}_1', \ldots, \underset{\sim}{X}_r')'$, where $\underset{\sim}{X}_k$: $n_k \times p$, $n_k \geq p$, $k = 1, \ldots, r$. Let $\underset{\sim}{W}_k = \underset{\sim}{X}_k'\underset{\sim}{D}_{n_k}\underset{\sim}{X}_k$, $k = 1, \ldots, r$ and $\underset{\sim}{W}_0 = \Sigma\underset{\sim}{W}_k$. The statistic is

$$t_4(\underset{\sim}{X}) = \prod_{1}^{r} |\underset{\sim}{W}_k|^{n_k/2} / |\underset{\sim}{W}_0|^{n/2} .$$

(5) The correlation coefficients. It is easy to see that the sample correlation coefficient between $\underset{\sim}{x}_i$ and $\underset{\sim}{x}_j$ can be expressed as

$$r_{ij} = \underset{\sim}{x}_i' \underset{\sim}{D}_n \underset{\sim}{x}_j / (\underset{\sim}{x}_i' \underset{\sim}{D}_n \underset{\sim}{x}_i \underset{\sim}{x}_j' \underset{\sim}{D}_n \underset{\sim}{x}_j)^{1/2} .$$

Let

$$t_5(\underset{\sim}{X}) = \underset{\sim}{R} = (r_{ij}) .$$

(6) The canonical correlation coefficients. Partition $\underset{\sim}{W}$ into

$$\underset{\sim}{W} = \begin{pmatrix} \underset{\sim}{W}_{11} & \underset{\sim}{W}_{12} \\ \underset{\sim}{W}_{21} & \underset{\sim}{W}_{22} \end{pmatrix} , \quad \underset{\sim}{W}_{11}: q \times q, \ \underset{\sim}{W}_{22}: (p-q) \times (p-q) .$$

The canonical correlation coefficients $t_6(\underset{\sim}{X})$ are the eigenvalues of $\underset{\sim}{W}_{12}\underset{\sim}{W}_{22}^{-1}\underset{\sim}{W}_{21}\underset{\sim}{W}_{11}^{-1}$. When $q = 1$, we get the multiple correlation coefficient.

(7) Testing the hypothesis that a covariance matrix is proportional to a given matrix. Assume $\mathrm{Cov}(\underset{\sim}{x}_{(1)}) = \underset{\sim}{\Sigma}$ exists; the statistic testing H: $\underset{\sim}{\Sigma} = \sigma^2 \underset{\sim}{\Sigma}_0 > 0$ is

$$t_7(\underset{\sim}{X}) = |\underset{\sim}{\Sigma}_0^{-1} \underset{\sim}{W}| / (\mathrm{tr}(\underset{\sim}{\Sigma}_0^{-1} \underset{\sim}{W})/p)^p .$$

(8) Testing the hypothesis that a covariance matrix is equal to a given matrix. The statistic testing H: $\underset{\sim}{\Sigma} = \underset{\sim}{\Sigma}_0 > 0$ is

$$t_8(\underset{\sim}{X}) = |\underset{\sim}{W}\underset{\sim}{\Sigma}_0^{-1}|^{n/2} \exp(-\tfrac{1}{2}\,\mathrm{tr}(\underset{\sim}{W}\,\underset{\sim}{\Sigma}_0^{-1})) .$$

(9) The generalized variance.

$$t_9(\underset{\sim}{X}) = |\underset{\sim}{X}'\underset{\sim}{D}_n\underset{\sim}{X}| .$$

Acknowledgment.

The authors wish to thank T. W. Anderson for assistance with this paper.

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4. TITLE (and Subtitle): Relationships among Classes of Spherical Matrix Distributions	5. TYPE OF REPORT & PERIOD COVERED: Technical Report
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Kai-Tai Fang and Han-Feng Chen	8. CONTRACT OR GRANT NUMBER(s): DAAG 29-82-K-0156
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Department of Statistics - Sequoia Hall, Stanford University, Stanford, CA 94305	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: P-19065-M
11. CONTROLLING OFFICE NAME AND ADDRESS: U.S. Army Research Office, Post Office Box 12211, Research Triangle Park, NC 27709	12. REPORT DATE: April 1984
	13. NUMBER OF PAGES: 23
14. MONITORING AGENCY NAME & ADDRESS (If different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Bartlett decomposition, elliptically contoured distribution, invariant statistics, uniform distribution on the Stiefel manifold, spherical matrix distribution.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Several classes of spherical matrix distributions have been studied by many authors. In this paper the relationship among the classes are discussed in the following aspects: the characteristic functions in these classes, coordinate transformations, marginal distributions, marginal densities, and sphericity. Some statistics that are invariant in these classes are listed for applications.